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Summer Research Project

The Use of Alignment in Ancient Near Eastern Mathematics

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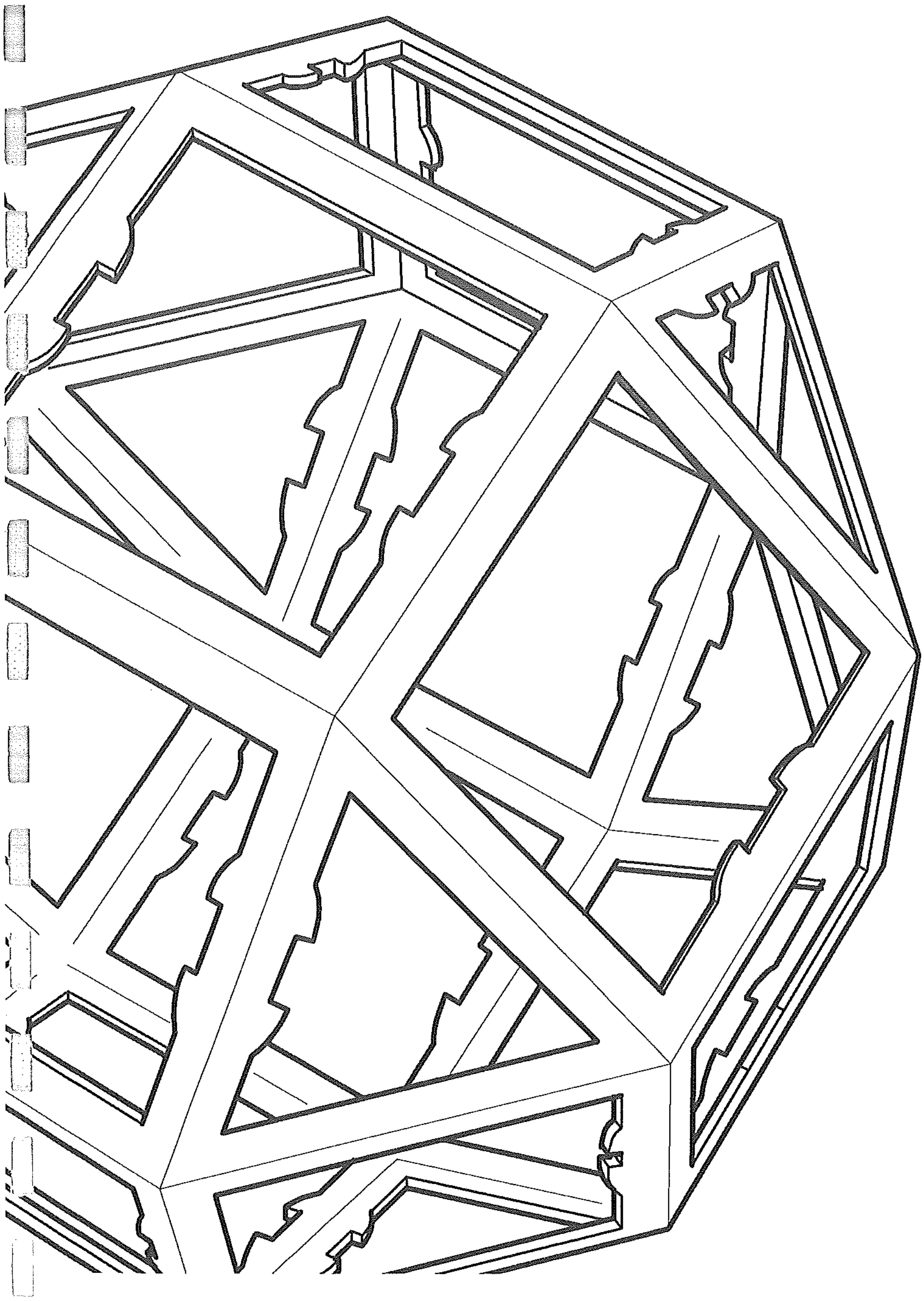

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The Use of Alignment in Ancient Near Eastern Mathematics

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Abstract

Cuneiform texts from the ancient near east include the oldest written mathematics in the world. These texts did not use symbolism to communicate mathematical operations, unlike modern mathematics. The hypothesis of this study was that alignment was used to perform a similar function. To investigate this, the study examined the formatting of different cuneiform tables, which inherently used alignment to imply an operation. A range of tabular development was included, from the simple, informal tables to sophisticated headed, complex tables. The study also looked at the use of alignment on 'rough working' arithmetical exercise texts, and identified two formats in which multiplication exercises were arranged. The study concluded that although alignment was used to show operations, this was a weak convention compared with the more prolific practices of left-justification, explicit statements through words and the concrete progression of logic from left to right along a line.

Contents

1 Introduction

The absence of symbolism is one of the first things that a mathematician will notice when studying cuneiform mathematics. Symbols are used extensively in modern mathematics to communicate the purpose of numbers. Cuneiform texts have a relative paucity of symbols to show the operation through which numbers are being manipulated. It is plausible that alignment might have served this purpose instead. This study investigates the use of alignment in mathematics from the ancient near east, and the extent to which it was an important communicative tool.

The first writing in the world, including the first mathematics, came out of the ancient near east - the land roughly between the Euphrates and Tigris rivers that is variously known as Sumer, Babylonia, Mesopotamia or roughly modern Iraq. Writing was created by making impressions in the clay tablets, orignially with accounting tokens and then with a wedge-shaped stylus by the mid third millennium BCE, hence the term 'cuneiform'. Also about this time the development of a place-value system that worked in base sixty, the sexagesimal place value system, allowed numbers to become abstract entities and mathematics to become an abstract activity (Robson, 2008).

This study looks at the use and importance of alignment and general layout within this mathematics. Intrinsically, alignment is about the placement and arrangement of numbers reflecting the relationship between them. When two numbers are aligned, that is, placed in corresponding positions vertically or horizontally, there is a visual communication that they are related. Cuneiform text did not have symbols to denote operations, such as the $\times$, $\div$, and $\cdot$ that we use today. The question then becomes whether alignment was used to perform a similar function - to indicate the operation, without the reader having to guess and work backwards to comprehend the text.This study is not inherently concerned with calculations themselves (and in many cases simply expands on the analysis of other authors), but rather focuses on the alignment and layout of those calculations.

The context in which ancient mathematics is studied is one of endless discussion. The argument here is that the physical alignment of the tablets is just as important as a sympathetic translation, constant referral to transliterations and an awareness of the social context in which they were written. Jens Høyrup and Jöran Friberg have both long stood in favour of translations that are as close as possible to the original semantics and syntax (Robson, 2007). Høyrup, and Otto Neugebauer before him, have both stipulated the importance of studying texts in transliterations rather than literal translations (Høyrup, 2002). The argument is that translations separate terminology and mathematical contents and thereby are mere "approximations" of the originals. More recently, Eleanor Robson emphasizes the importance of studying texts in their mathematico-historical context (Robson, 2001), arguing that mathematics is the product of individuals, and individuals are the product of the society in which they live, work and think (Robson, 2008). It is the intention of this study to add another facet to this mix, and say that the examination of the visual traits rather than just numbers and words themselves may lead to a more insightful and perceptive understanding of ancient near-eastern mathematical texts.

Alignment is not something that many modern scholars have considered in its own right. Jöran Friberg has made a by-the-by attempt to classify different formats for arithmetical table texts, and in places alludes to the 'look' of texts (Friberg, 2007), but does not make an analysis of the significance of these formats. Eleanor Robson has written a number of papers specifically looking at the development and use of ancient Mesopotamian tables, and discusses the meaning these in terms of conceptual understanding (Robson, 2004)(2005). She also makes mention of patterns of number arrangement (Robson, 2007), but does not elaborate on the implications of these. No one has taken a broad look at the consistencies and inconsistencies of the arrangement of different types of cuneiform text and the importance of such.

The intention of this study is to get an overall appreciation of the use of alignment in ancient near-eastern mathematics. Tables are a format that inherently express operation through alignment. Though the nature of that operation may be clarified by column headings (in modern tables), the existence of a relationship is understood through the arrangement of data into columns and rows. We understand that visual correspondence signifies mathematical correspondence. In looking into the conceptual relevance of alignment, this study attempts to classify the different formats seen within tables, and comment on their use of alignment. There will also be comparisons of simple and complex tables (see §2.1). Within this will be the issue of whether words were ever replaced by alignment, either by being condensed into headed tables or being omitted altogether.

Arithmetical exercise texts (rough calculations or pedagogical exercises) are tablets that have the potential to be more random in their alignment of numbers. It is for exactly this reason that the absence of symbols such as $\times$ and $\div$ is more noticeable, and the question of the use of alignment becomes more pronounced. Also on these tablets, the study considers the possibility of evidence that scribes used alignment to remind themselves of the relative places values, given that the sexagesimal place value system gives no indication of the actual order of the numbers, only of the relative value of the places. The study will also consider the use of visual features, such as ruled lines and, conversely, of gaps to accentuate alignment.

This study will be looking at evidence encompassing texts from Ur III period in the later third millennium BCE to the Neo-Babylonian Period of the early first millennium BCE, with a large portion from the Old Babylonian Period of the early second millennium. Transliterations and translations tend to approximate the alignment of figures and do away with visual elements such as ruled lines. Therefore, it is vital to study texts from photographs and drawings rather than from transliterations or translations, although both will be taken into consideration. The aim was to study a representative range of extant tablets, including arithmetical exercises, pedagogical documents, and tables both formal and informal. Texts will be largely numerical in nature, although the layout of a certain number of mathematical word problems have also been added for the purpose of discussion, as will a brief mention of tabular accounts. Otherwise, accounts or other applied mathematical documents will not be looked at, no will diagrams, such as field plans or geometric problems.

In this study, alignment that is vague or inconsistent and may even be accidental is termed *non-rigid*, whereas *rigid alignment* refers cases where numbers are in straight lines or are repeatedly placed in exactly corresponding positions.

The mathematics with which this study is concerned used a sexagesimal place value system (SPVS). In modern base 10 end zeroes and decimal points are used to give an indication of the absolute order of magnitude. In contrast, the SPVS only indicates the relative magnitude of places within a number. In some younger texts a sign is used to indicate an empty sexagesimal place but there never any 'end zeros' to show where the fractions begin. Any number could be to any order of sixty and be no different for it. For example, 2 30 might mean $2 \times 60 + 30 = 150$ or $2 + \frac{30}{60} = 2\frac{1}{2}$, or any multiple of any order of 60 thereof. At times where the discussion requires that an arbitrary or convenient order is chosen or guessed, I use the generalized form of angle notation that was first used by Assyriologist F. Thureau-Dangin (Høyrup, 2002). Hence, $3°15' = 3\frac{1}{4}$ and $3`15° = 3 \times 60 + 15 = 195$. Otherwise, I simply leave spaces between sexagesimal places, remaining ambiguous about the absolute value of each place (for further discussion of SPVS, see §3.1.2).

2 Tables

2.1 Definitions

Mathematical tables in the ancient near-east pre-date cuneiform culture (Robson, 2005), but they were never, by any means, a prolific format. Nevertheless, as inherently 'aligned' texts and as excellent indicators of the conceptualization of alignment, they have an important place in this discourse.

While there is plenty to discuss about tables in the ancient near east, it is first necessary to define what a table is. This is a topic about which there is no little debate. Dominique Tournès (2011), one of the organizers of a project to publish on the significance and use of historical numerical tables, defines a table as "a list of lists of numbers having same length n ($n > 2$), when this list is used to express a correspondence between n sets of numbers ...". He stipulates that this is distinct from the arrangement of numbers intended to "facilitate the application of an algorithm"; therefore, ruled tablets such as MS 2728 (Fig.18) are not tables despite having visual rows and columns. Also, he distinguishes between 'tables', which show only one function (if several independent variables or 'dimensions'), and 'arrays', where multiple sets of data are arranged into tabular form for presentation and analysis. It is also Tournès' opinion that tables can be purely textual or lists of lists presented in a linear format. He notes that these forms are particularly prolific in ancient times, and this is a valid statement true. By Tournès' definition, the majority of the texts discussed are linear tables and a handful of arrays.

This definition is a solid one, but it places the importance of function slightly above the criticalness of alignment. When Eleanor Robson (2004)(2005) studied tables in the context of Ancient Mesopotamia she took Tournès' 'arrays' to be proper tables and characterized linear tables, which use alignment only in its simplest form, as 'numerical lists'.

For this report, where alignment is the focus but the sophistication of the table is also critical, a combination of these two approaches is called for. Here, a table is list of lists where there is a defined relationship (in modern terms, 'function') between one column and another. These relationships may be successive (as in, column a determines column b which determines column c) or all columns may be determined by an initial column (as in, column a determines b and c but there is no direct relationship between b and c).

In any case, the vast majority of 'tables' from ancient Mesopotamia have only two columns (a variable and a result). They are, as Tournès would say, one dimensional, *i.e.* they have one independent variable. For clarity, table such as these, which are governed by only one function, are defined here as *simple* tables.[1] *Complex* tables have three or more columns and the functions between the columns differ. There are very few examples of these but as substantial leaps in the use of alignment on a large scale they extremely interesting.

Robson (2004)(2005) puts further definitions on tables which are also employed here. She classifies tables that are ruled horizontally and vertically as *formal* and tables that use spatial

[1] I suppose by this definition I would regard all of what Tournès classes as tables as being 'simple', even if they had two or more independent variables. However, there are no such tables known from the ancient near east.

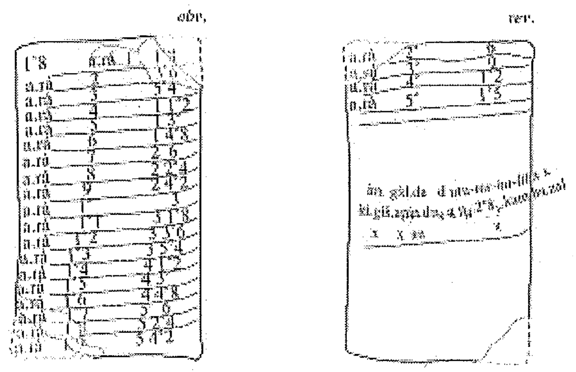

Figure 1: MS 2708, a format one, 18 times multiplication table. (Friberg, 2007)

arrangement only as being *informal*. She also the clarifies that tables with columnar headings are *headed* tables; these are of critical interest in §2.3.

Although there is a reasonable corpus of extant tables, this study is solely concerned with texts that appear to be 'purely' mathematical, even if they were studied with very practical purposes in mind. Hence metrological tables or accountancy tables are not covered here.

2.2 Simple Tables

2.2.1 Single Tables

Extant simple cuneiform tables are largely paedological, used to assist the memorization of multiplication, squares, reciprocals, and square and cube sides (Robson, 2005). Although they are regularly called tables, their perpetual use of words in fact means that they are better characterized, as Robson (2005) does, as 'numerical lists'. Although they are sometimes ruled on every line, they are almost all informal tables (the few exceptions are dealt with in section 3.2).

There are three formats that known multiplication tables follow. In the most common, here called format number one, the first line of this format reads as it would be spoken ('a.rá' is the transliteration of the cuneiform symbols for 'times', or more literally 'steps of' (Robson, 2008)):

[multiplicand] a.rá 1 [product]

The next lines drop the multiplicand and simply reads:

a.rá [multiplier] [product]

MS 2708 (Fig.1) is a classic example. The first line reads 18 a.rá 1 18, with a gap separating the multiplier 1 and the product 18. The next line omits the multiplicand 18 and instead the 'a.rá' is left-justified rather than being indented. This statement of the multiplicand in the top line only reduces repetition but the left-justification destroys the alignment of the columns and means that the first line comes out of sync (see §2.3).

The second format reads much the same but repeats the multiplicand on every line, as in MS 2184/3 (Fig.2). This preserves the vertical alignment and keeps the data in definable columns, yet it misses the potential for alignment to serve instead of repeating the multiplicand.

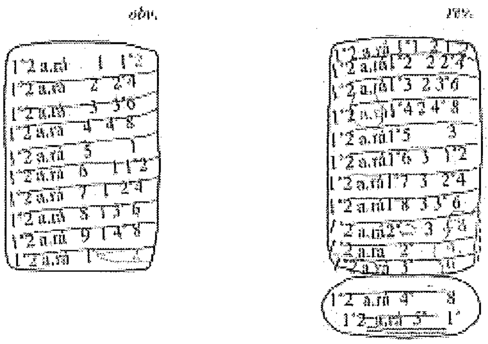

Figure 2: MS 2184/3, a format two, 12 times multiplication table. (Friberg, 2007)

Figure 3: MS 3044/3, a third format, 45 times table. (Friberg, 2007)

Format number three is the only genuinely tabular type among the simple tables. It omits words all together and simply has two columns of numbers (the multiplier and the product), usually at the extreme sides of the tablet (see MS 3044/3, a 45 times multiplication table (3)). Tables drawn this way tend not to state the multiplicand or the function at all, meaning that an outside reader has to work backwards to discover what the table is actually saying.

Tables for other functions can broadly be seen to fit into these formats as well. Tables of reciprocals follow format number one - they state the '1' (or '60) at the start of the top line, which continues as all the other lines read as:

$$\text{Its } n \text{ part is } \frac{1}{n}$$

Format one is incompatible with tables like squares and square roots (here called 'square-sides' to reflect the way they were conceptualized by the author) where the operation is consistent but there is no single constant number (*i.e.* the multiplicand for multiplication tables). These tables are most common in format number two. MS 2794 (Fig.4) is a typical example of a square table. Much in the manner of squaring exercises, number being squared is repeated twice for each line. Tables of square sides (square roots), cube sides, and similar[2] only give values for integer roots. A standard line will read:

$$n^p \text{ e } n \text{ íb.si}_8$$

where p is 2 or 3, or alteratively for square sides tables:

$$n^2 \text{ .e } n \text{ ba.si}_8$$

MS 2996 (Fig.5) is a cube sides table that shows a slightly different format, which is essentially a variation of format number 2. Its lines read:

[2]Tables for quasi-cube sides, which give n for $n \cdot n \cdot (n+1)$, or for $n \cdot (n+1) \cdot (n+2)$ are also know. There may also have been similar tables for $n \cdot n \cdot (n-1)$ (Friberg, 2007) but none are extant.

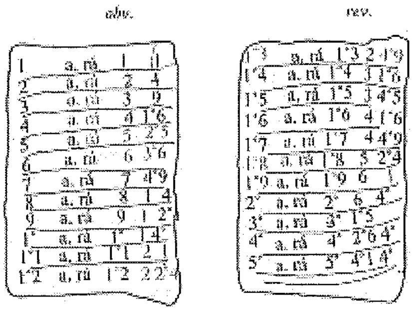

Figure 4: MS 2794, a format two table of squares. (Friberg, 2007)

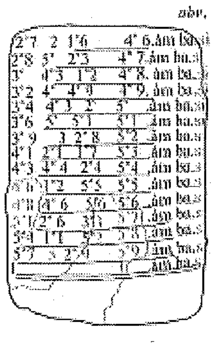

Figure 5: MS 2996, a table of cube roots, in a variation of format two. (Friberg, 2007)

n^3 n .àm ba.si

The magnitude of cubes means that tables of cube sides tend to get cramped and messy. MS 3863 (Fig.6) is one example of a tablet where it is had to distinguish the numbers, despite the dividing '.e'. MS 3973/1 (Fig.7) and MS 3966 (Fig.8) are both examples of format three cubic tables. There are no known examples tables of squares[3] or reciprocals in format three.

While all such tables show 'tabular' alignment to a certain degree, in many cases the alignment is non-rigid, both horizontally and vertically. Horizontally, the rows may be buckled, curved or slanted. Often, line rulings have to make up for any discrepancies. Where line rulings are absent, rows can become extremely unaligned. One extreme example is the reverse of MS 2877 (Fig.9). Lines are artificially drawn in on the left-hand side to illustrate how unaligned the rows become. On the original on the right, it is impossible to realize the mistake until a close inspection is made. There are also many examples of poor alignment vertically. MS 3870 (Fig.14) is one text were the 'column' of the multiplier curves about. Robson (2005) also notes that columnar constraints are often broken when convenient. Simple tables such as these show that the use of rudimentary alignment was common, but it was not particularly stringent or formal.

2.2.2 Table Compendiums

'Table compendium' is the term used here to describe tablets that are a collection of simple tables. They range in size from double tables such as MS 3964 (10) to the large MS 3845 Fig.11

[3]Actually, this is debateable - see section 3.2.

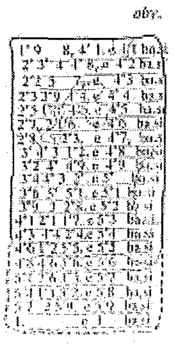

Figure 6: MS 3863, a table of cube-roots where it is had to distinguish between the 'columns'. (Friberg, 2007)

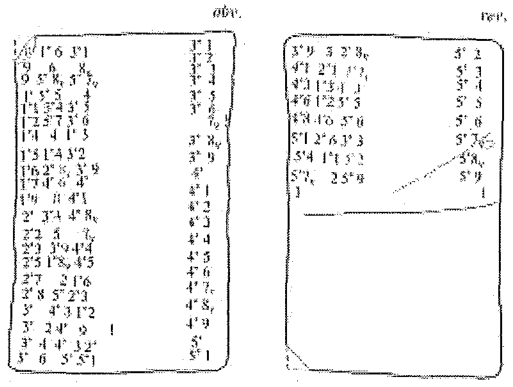

Figure 7: MS 3973/1, a format three table of cube-roots. (Friberg, 2007)

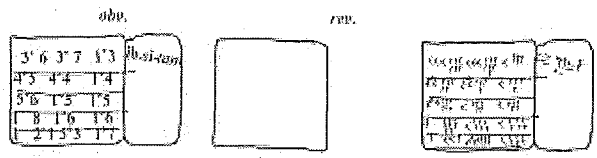

Figure 8: MS 3966, a small, format three table of cube-roots. (Friberg, 2007)

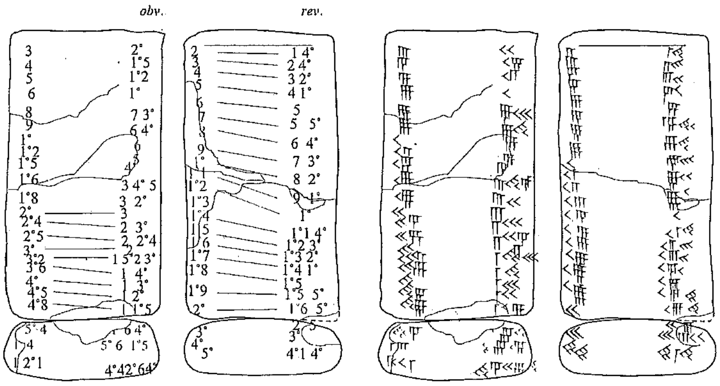

Figure 9: MS 2877, a double table showing reciprocals and 50 times tables. Note the huge ill-alignment on the reverse. (Friberg, 2007)

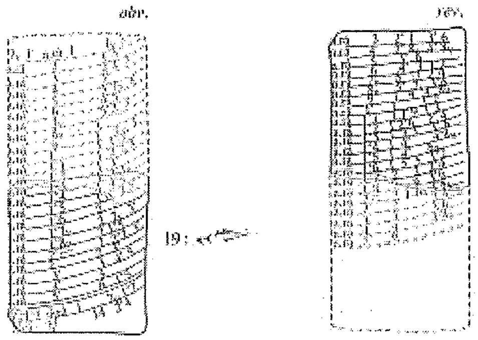

Figure 10: MS 3964, a 'double table' - a table compendium containing two individual tables. (Friberg, 2007)

and the huge MS 3974 (Fig.12, which contain eleven and forty individual tables respectively. These works are too big and complicated to have been simply for memorization - it is more likely that they were a teacher's 'master-copy'.

The individual lists in table compendiums appear in the same formats as single tables, but occasionally they switch formats (usually between formats one and two). This usually happens from one individual table to the next. Interestingly, table compendiums rarely clearly distinguish between individual tables in any other way, despite their probable archive status. In a similar way to the double lines that are often seen at the ends of single tables, MS 3939 shows double lines between its eight times and 7 30 times sections (Fig.), and also on MS 3870 at all the list changes. Yet neither of the very big table compendiums (MS 3845 and MS 3974) show any divisions between the individual tables. Given the amount of information on these tablets, this lack of clarity may seem confusing. It seems especially odd in comparison with the clear divisions shown in other text, such as triangular multiplication exercises that are ruled between each problems, or the distent lines separating different sections on the word-problem tablet AO 6484.

That these tablets lack clarity is also significant on a more conceptual level. Although table compendiums show tables for more than one function, they are not complex tables. This is because they are not arranged with multiple numeral columns with the relationships working

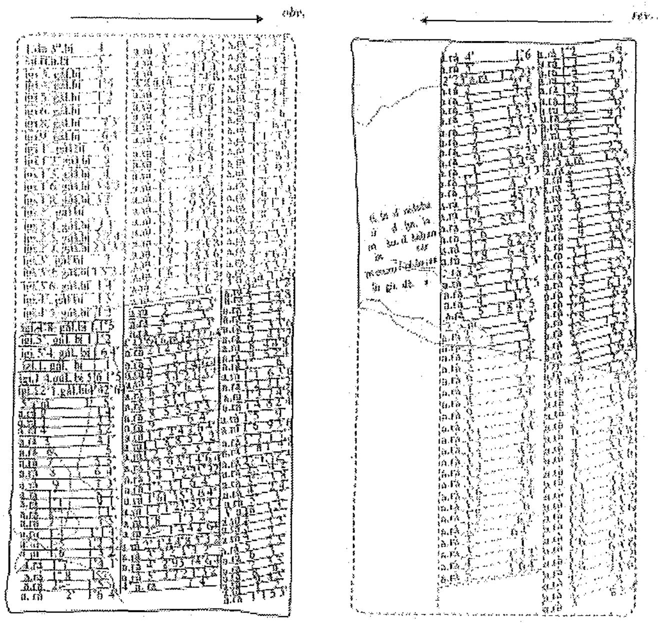

Figure 11: MS 3845, a large table compendium containing eleven individual tables. (Friberg, 2007)

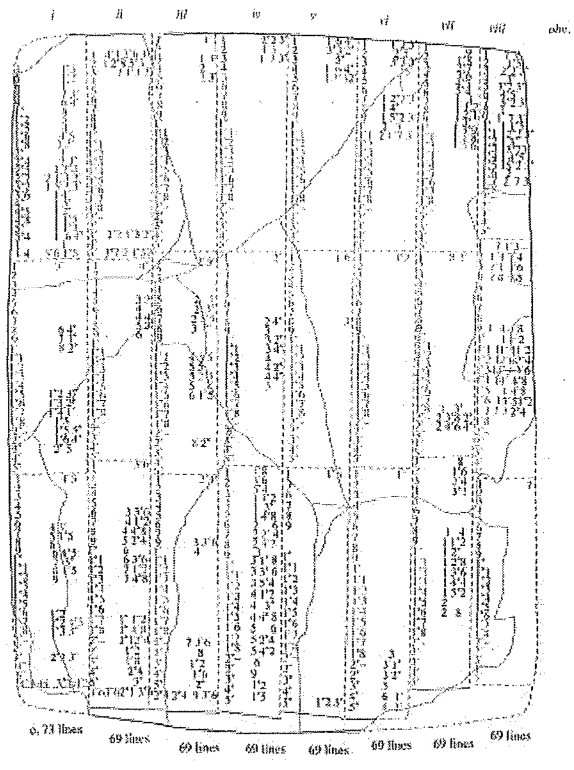

Figure 12: MS 3974, a huge table compendium containing forty individual tables. (Friberg, 2007)

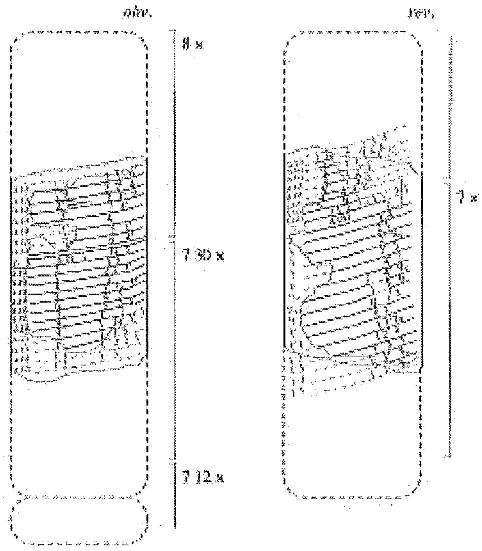

Figure 13: MS 3939, a fragment of a a table compendium featuring double lines between individual tables. (Friberg, 2007)

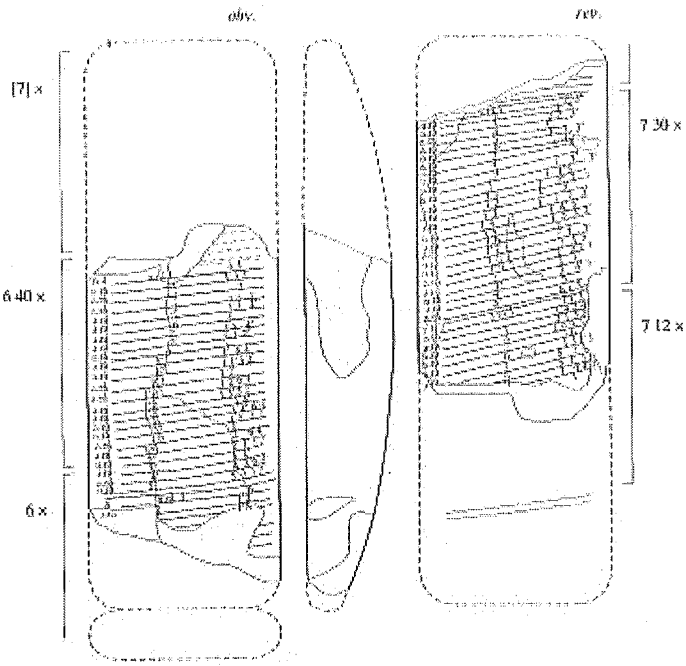

Figure 14: MS 3870, a table compendium featuring double lines between individual tables and a wobbly multiplier 'column'. (Friberg, 2007)

between the columns. Instead, the lists follow on from each other through multiple narrative columns. This is symptomatic of an underdevelopment of the concept of tabularization and communicating through alignment. These are tables that had the potential to be written as complex tables, with the information of each individual list expressed in a single column. That they were presented instead as rather messy and laborious compendiums is an indication that, while the material step of re-arranging a fresh clay tablet is easily done, the cognitive step of placing each set of output values in a single column is more complicated than the modern mind gives it credit for, and that it had not occurred in the ancient near-eastern mindset.

2.3 The Use of Words

In modern numerical tables words and labels are confined to titles and column and row headings. Readers are used to these single explanations applying to multiple cells and explaining not just each cell but the table as a whole. This condensed form of presentation is not at all common in cuneiform texts. In fact, when debating how 'tabular' cuneiform tables are, it is impossible to avoid the frequency with which words appear *within* tables. As seen in section 2.2, in the vast majority of extant tables the relationship between the columns is explicitly written *for every line*.

Consider the use of words and repetition in each of the three formats. The first format has an almost 'title-like' first line that serves to explain all the lines beneath it. But the words 'a.rá' is repeated through all the subsequent lines, so the alignment in these tables is not highly important to the understanding of these texts. It is this feature that means that they are best described as 'lists'.

The same can be said, and even more so, for the second format. Although the repetition of the multiplicand on every line prevents the sort of left-justification that disturbs the vertical alignment in the first format, that constant and complete repetition to not in any way allow alignment to act as a communicative tool. This is perhaps explained by the fact that these text were largely used for memorization, so writing out each line as it would be spoken possibly makes sense. In fact, the only alignment that could be said to be at play in either of these formats is the word-less gap between the multiplier and the product. Yet the absence of a line in this area shows that the ancient near-east authors did not wish to interrupt the prosaic flow of their text. In the vast majority of cases, it seems that adherence to language outweighed visual impact.

Format three shows a difference in this respect. In omitting words from the body of the table, this format uses alignment to implicity express what words usually make explicit. In this respect, format is actually the only tabular format that have been looked at under 'Simple Tables'. Yet while the existence of a function is made apparent through alignment, the absence of columnar headings or an overall title makes the actual function a mystery (although it may quickly become apparent through inspection). So while there are tabulation elements that are used to express the operation of the table, there is no combination of the communicative powers of language *and* alignment. This evidence suggests that there may have been a certain 'all or nothing' attitude towards prosaic and tabular formats. An indisposition to combine the two meant scribes never harnessed the potential of tables to express mathematical tables succinctly.

This observation makes Plimpton 322 an incredible exception. Plimpton 322 is the only extant headed table from the entire corpse of ancient near eastern texts, and indeed the only truly tabular text from ancient Mesopotamia. For further discussion, see section 2.4.

2.4 Complex Tables

Extant, cuneiform, complex, mathematical tables are exceedingly rare from the ancient near east, to the extent that the corpus is dealt with here in its entirety.

Ash 1924.796 (15) is a gorgeous text from sixth century BCE Kish (Robson, Katz), and was probably meant for library archives (Robson, 2007). It is, as (Robson, 2005) says, "the only identified table that combines both squares and square roots in a single entry". The key word here is that it *combines* two functions into a single row. This is a gigantic step towards using tabulazation to succinctly express operations and inter-connected relationships. But this potential is not quite realized here. Rather than the table working two ways and [$n \quad n^2$] being understood to communicate both the square of n and the square-side of n^2, the lines linearly read:

$$n \text{ times } n \qquad n^2 \text{ the square side is } n$$

It still repeats phases to express the operation, and in Robson's opinion (2005) "The lines of the tabular formatting appear to have served more as aids to producing a beautifully written document than as conceptual separators of different classes of data.". Ash 1924.796 is undoubtably at the top end of Ancient near-eastern tabularisation, but is not quite a true table.

There is in fact only one known text that shows a truly tabular understanding as it is considered today. Plimpton 322 (Fig.16) shows a sophistication that is quite remarkable in light of the many pseudo-tabular texts that have been examined in this study. Broken on the left-hand side, the extant fragment shows four columns of fifteen rows. The right-most column is a row count. Partly because of the probable loss of at least one column on the left (and likely more), there is much debate as to what it actually shows mathematically. Alone of all the extant, mathematical texts from Ancient Mesopotamia, its columns are headed.

Robson (2005) emphasizes the similarities between Plimpton 322 and the many administrative tables that came out of Larsa in the second decade or so of the 1700s BCE. (Larsa was the providence of origin given by the original dealer.) Robson points out the heading of the final column is 'MU.BI.IM', meaning 'its name' (*i.e.* the name of the row) is the same as the standard practice in administrative tables. In Robson's opinion, the landscape orientation is also significant, as mathematical tables were typically portrait orientation, but accountancy tables of that era were far more likely to be landscape (Robson, 2004).

There are far more tabular texts that are administrative records rather than primarily mathematical. The development of tables for administration was fitful and inconsistent, but there in certain eras there was a significant output of distinctly tabular temple accounts and live stock records. These documents tend to show formal tabularization, rather than the informal and prosaic lists that come from the paedological corpus. They were never a prolific format, but at some times evolved into acceptable, formal records (Robson, 2004). Robson (2004) concludes that tables were a matter of personal preference for accountants. They were undoubtedly were a more efficient and "powerful" form of book-keeping, which allowed for easy calculations and data storage (Robson, 2004).

These documents would definitely be classed as arrays by Tournès. The multi-faceted nature of the data and its layout for calculations and analysis is not what he would classify as purely tabular. Nevertheless, such documents show significant cognitive processes and indicate that there was the potential for the use of tabularization in pedological documents.

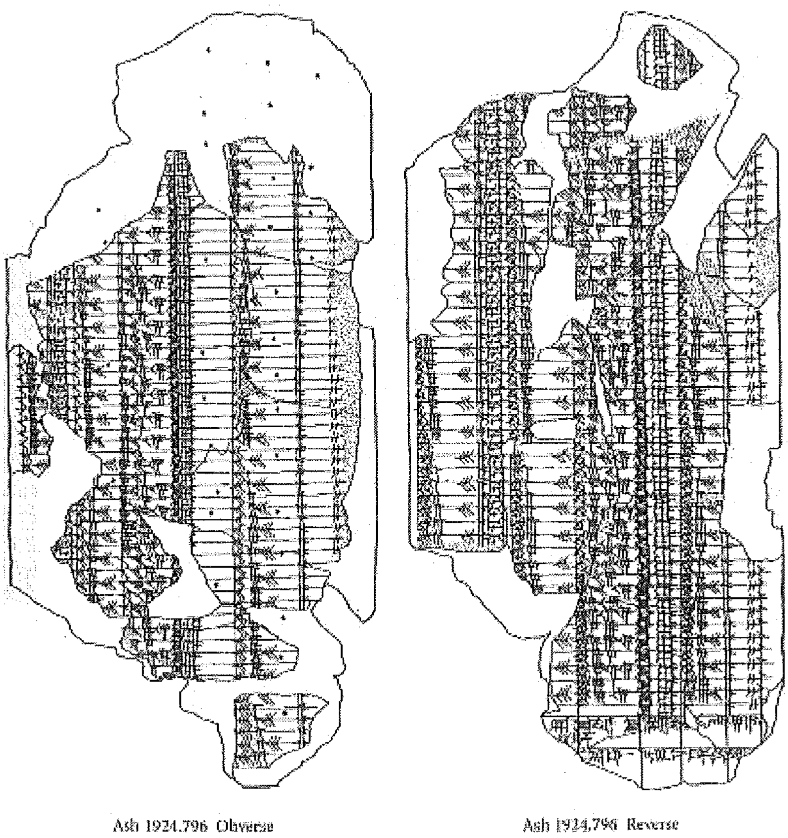

Figure 15: Ash 1924.796, a complex table that goes up in intervals of 30. Shows the square and the square-root of the square in the same line. (Robson, 2007)

Focusing on Plimpton 322, it is undeniable that the columns are far more complex than anything else that is purely mathematical. Whatever the table actually shows, the amount of information and the distinct, multi-faceted relationships between the columns are unlike anything else from the 'pure' mathematical corpus. The 'array' nature of Plimpton 322 is certainly something usually seen only in administrative documents.

If Plimpton 322 is considered from this perceptive, it was not so much an evolution from simplistic paedological tables to fully-fledged mathematical tables, but more of a 'cross-over' of two formatting approaches that were usually kept separate.

3 Arithmetical Exercise Texts

3.1 Numeral Alignment

3.1.1 Operation Indication

Arithmetical exercise texts are prime sources in the search for evidence of the use of alignment to indicate operation. These tablets are largely pedagogical and show arithmetical exercises such as multiplication, squaring and exercises involving division. There is some debate as to what level of learning these tablets represent. Friberg considers them to be the product of beginners, were as Proust would largely classify them as advanced (Friberg, 2007). Regard-

Figure 16: Plimpton 322, a complex, headed table. The extant fragment shows four columns and fifteen rows. (Robson, 2005)

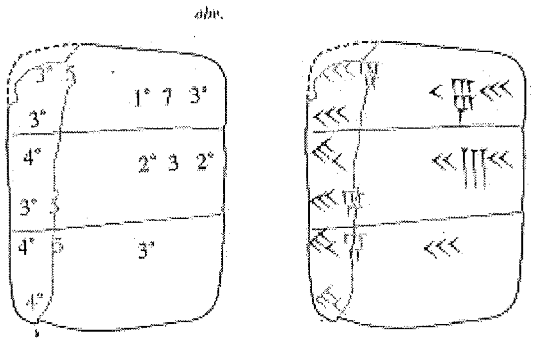

Figure 17: MS 2729, a tablet ruled horizontally between triangular multiplication exercises. (Friberg, 2007)

less, they were not designed for archives or presentation, and written (presumably) by many different authors. Therefore, any consistencies in alignment gain new significance, as they may suggest at something approaching a convention.

On such tablets, there are two formats that multiplication can be seen to follow. The first is a triangular arrangement, where one factor is directly below the first and the product is placed to the right. The product is usually aligned between the factors (forming an isosceles triangle), such as in the second line of MS 2729 (Fig.17) where 40 is multiplied by 35. It can also be closer in line with the first factor (forming a right-angled triangle), for example in the second line of MS 2728 (Fig.18) where 55 is multiplied by 50. Occasionally it is seen to be far higher than the factors, as it is on the last line of MS 3045 (Fig.19).

This 'triangular multiplication' arrangement, as the format is called here, is usually seen on tablets where multiplication is the only operation being completed. It does not appear to be a rigid format; occasionally, where the product is too long to fit next to the factors, it is placed directly below them, for example on CBS 3551 (Fig.20). Squaring exercises consistently follow the triangular format (in Old Babylonian calculations, the number to be squared is always stated twice). Occasionally, a continuous form of triangular multiplication can been seen. This is where one product is then used in another multiplication. The first calculation is set

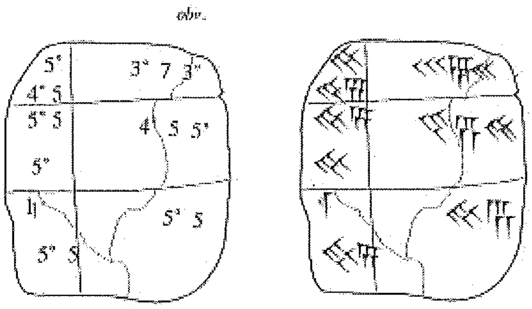

Figure 18: MS 2728, a tablet showing triangular multiplication that is ruled horizontally and vertically. (Friberg, 2007)

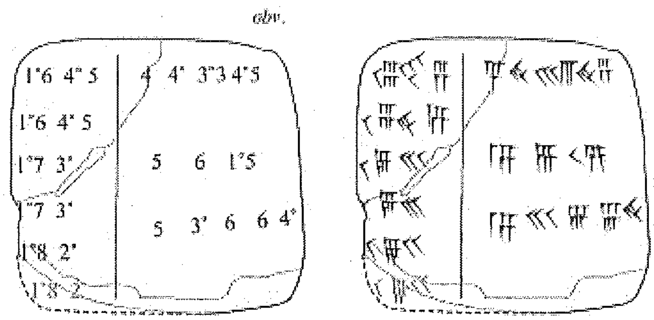

Figure 19: MS 3045, a tablet showing triangular multiplication that is ruled vertically between factors and products. In the final equation, the product is far out of line with its factors. (Friberg, 2007)

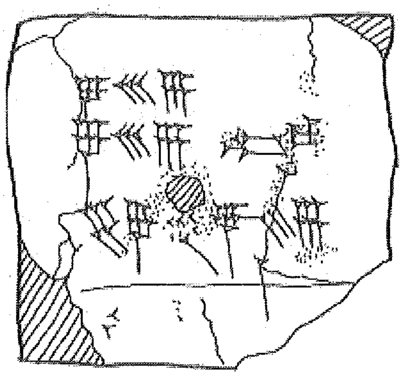

Figure 20: CBS 3551, a tablet showing multiplication where the product has been placed below the factors. (Robson, 2008)

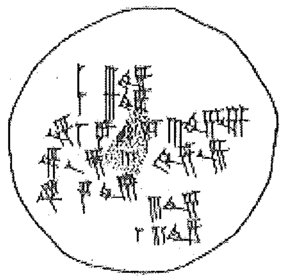

Figure 21: UET 6/2 222,

out in standard triangular form, then the third factor is placed beneath the first two and the second product is placed to the right of this number. UET 6/2 293 (Fig.23)and UET 6/2 236 (Fig.24) are two examples of this. The use of pre-drawn lines[4] on tablets such as MS 2728 suggests that this was a format that a student was instructed to follow, rather than one that occurred accidentally.

Multiplication is also seen in another format, which can be characterized as 'straight multiplication'. Here, the factors are placed side by side on the same line. and the product is written beneath them. There is often very little visual separation between the product and usually none at all (see §3.3). It is predominately seen on tablets where other calculations are being performed or where the multiplication is a single step in a longer exercise. On UET 6/2 222, (Fig.21) for example, it can be seen within the third line that 1 7 44 3 45 is immediately followed by 16 and the two numbers are multiplied to give 18 3 45 on the next line, which is also followed by 16 and multiplied by this to give 4 49.

Division is a harder operation to comment on, as ancient near eastern scribes typically completed division by finding the sexagesimal reciprocal of the divisor and multiplying it by the dividend (Robson, 2008)). Reciprocal problems themselves appear to show an alignment convention, although simple reciprocals are somewhat messier. On tablets where finding the reciprocal is the object of the exercise itself, the reciprocal is usually seen on the line below the original number. MS 2730 (Fig.??MS 2730)) is one such tablet; here, 4 51 16 16 is

[4]It is possible to say that they were pre-drawn as they divide the tablet in to roughly equal portions and in places the scribe slips over them, rather than the lines being fitted around the text.

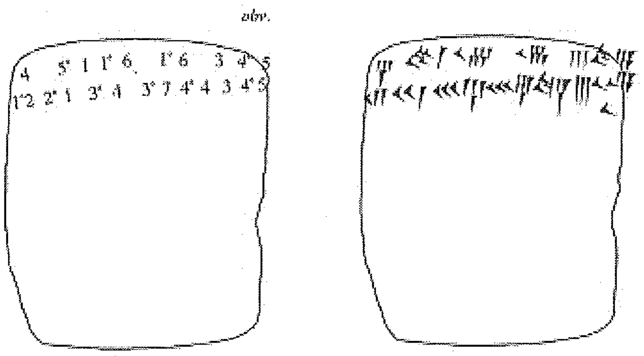

Figure 22: MS 2730, a tablet showing a reciprocal exercise.

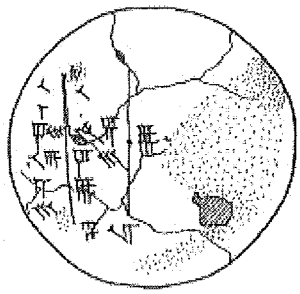

Figure 23: UET 6/2 293, .

written at the top of the tablet and its reciprocal 12 21 34 37 44 is written on the line below it. (According to Friberg (2007), the 3 45 that appear at the ends of both lines are residual numbers from the 'trailing part algorithm that he suggest was used to find difficult reciprocals such as these.)

At other times, simple reciprocals that the scribes would have memorized are placed randomly next to the number they are the reciprocal of and then used in division calculations. UET 6/2 293 , for example, is a hand tablet that shows a series of calculations, starting with:

10
(×)1(=)10
(×)4 30 (=)45

When the author reaches this point and wishes to divide by ten, he has written the 10 and placed its reciprocal 6 directly on its right, then calculated the answer, 4 30. In the next step, where 4 30 is divided by 30 to give 9, the reciprocal 2 is located directly *above* the 30.

UET 6/2 236 is a very similar tablet with comparatively inconsistent alignment: it starts with:

50
×30 = 25
×3 = 1 15

When the scribe divides 1 15 by 10 to get 7 30, and the reciprocal 6 is written just after the 10 on the right. On the next line, in dividing 7 30 by 30 to get 15, the reciprocal 2 is placed just before the 30 on the left. This kind of inconsistency suggests that there was no formal convention for the placement and alignment of reciprocals.

Considering that scribes were discouraged from showing intermediate working (Robson,

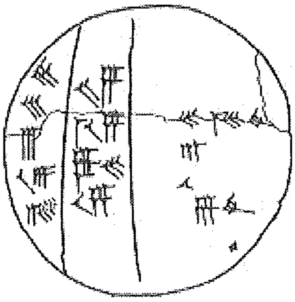

Figure 24: UET 6/2 236, a tablet showing continuous triangular multiplication, including the placement of simple reciprocals. (Robson, 2007)

Figure 25: MS 2317,

2008), it is possible that the reciprocal was not meant to be written at all and that the scribe was technically meant to carry out the whole procedure (finding the reciprocal and then performing the multiplication) is his head.

The few direct division exercises that were found in this study show no consistent format. MS 3871 (??) is one tablet that shows a direct division exercise. It shows no intermediate working, but simply reads:

4 37 46 40
11 34 26 40
2 30

As 4 37 46 40 times 2 30 equals 11 34 26 40, the format here is divisor over dividend over quotient. This is contradicted by the layout of MS 2317 (25), which Friberg interprets as the division of 1 1 1 1 by 13 to give 4 41 37. Here the format is dividend over divisor over quotient.

Other calculations involving division do not suggest whether one of these was the standard. Long factorization problems such as MS 2242 (Fig.27) and MS 3037 (Fig.??) could be said to follow either format as they do not show the divisor every at step. Sole reciprocal problems (like MS 2730 (Fig.22)) are equally ambivalent. The likes of UET 6/2 293 and UET 6/2 236 show no such line-by-line arrangement and the presence of reciprocals explains why they are more consistent with the multiplication arrangements that have already been discussed.

It is necessary to consider these were rough working tablets, the equivalent of modern margin scribbles. It is likely that the author of such calculations did not intend them to be used again, even by himself. Therefore, the scribe may not have felt it necessary to indicate the operation at all. Similarly, it is perfectly conceivable that the author was free to break with alignment conventions and place the numbers in what ever way made sense in his own head.

Given the evidence, it seems that there were some conventions by which multiplication was indicated through alignment. Any corresponding practice for other operations is not apparent; it appears that calculations were either transformed into multiplication or merely arranged as space allowed. However, these arrangements are generally only seen on rough-worked exercise tablets.[5] More often than not, problems run in straight lines (from left to right) and words have been used to clarify the operations (for further discussion, see §2.3). This is notably true for pedagogical tables and other documents that were likely to be kept and refereed to. The conclusion then is that while alignment was informally used to show operations, it was not a rigid format.

3.1.2 Relative Place Indication

Although the SPVS is called a place value system, it is not so in one respect: the absence of end zeroes or decimal points means that it does not show the absolute value of the places, only the relative order of places within a number. Yet often numbers did have an absolute value, if only in the scribe's mind. The development of the SPVS allowed numbers to become abstract for the first time, but they could be turned back into properties of real world objects when the calculation was complete and appropriate units were re-applied (Robson, 2008). In calculations that involved units it was necessary for scribes to remember the place values of each number that was affecting the equation (Neugebauer, 1970), and mistakes occurred when they were not (Robson, 2008). The possibility then arises that alignment might have been used to communicative what the SPVS did not: the relative place value *between* numbers.

Tablets that show units make it possible to investigate this. YBC 1793 (Fig.26) is a tablet that shows additions of silver weights - it is the earliest dateable sexagesimal scratch calculation (Robson, 2008). The first four lines on the left-hand column read:

14 54
29 56 50
17 43 50
30 53 20

The total of these numbers expressed in units of silver is $1\frac{1}{2}$ minas, $3\frac{1}{2}$ shekels minus 7 grains of silver, and this is written on the tablet on the line before the numbers. Here, the relative places appear to be vertically aligned (although the 53 20 in the last line is slightly askew). However, as the numbers are also left justified it is impossible to state categorically that this was done to ensure the addition of the correct places. The more intriguing addition problem is in the middle of the right-hand column, which reads:

2 54
   45
   28
   17
2 28
   27

Again these are added, with the total in silver written below. The arrangement here is not nearly as orderly as the first example. There is a heavy leftward curve in the numbers on the right and in several places there non-relative places are in line. Nevertheless, it is significant that the numbers are not quite left justified, particulary as the final number, 27, seems to be aligned with the 28 of the 2 28. This is a possible indication that scribes used alignment to assist in calculations and keep track of absolute value of each place, even if they were not very diligent about it.

[5] For discussion of the two possible exceptions, see section 3.2.

There are many extant rough-working tablets and pedagogical exercise tablets where units were not stated at any point. In these cases there is no way of knowing what the absolute place value of a digit is. Fortunately, in some texts where a single root is being manipulated an arbitrary absolute value can be applied and the alignment of relative place values can be investigated.

MS 2242 (Fig.27) is one such tablet in which a single constant value is used over multiple lines. It shows the factorization of 46 20 54 51 30 14 03 45 into its sixth root 3 45. If the root 3 45 is (arbitrarily) taken as $3°45'$, then it can be seen that the 'degrees' place is placed more or less along a line while the fractions of 60 splay out to the right. Certainly, the numbers were not left justified, which would have destroyed all relative connection between the place values. However, the alignment is not perfectly straight and could almost be accidental.

This evidence for the alignment of relative place values is contradicted by MS 3037 (Fig.28). This is an Old Babylonian tablet that shows 3 11 6 10 42 48 57 36 successively divided by 12, right down to 2 24 (which is 12^2). It is effectively a factorization problem that shows that 3 11 6 10 42 48 57 36 is the twelfth power of twelve. Based on this, the assumption is that each step is a division of $12°$. But in this case, no matter what absolute order assigned to the starting value, the relative places run on an angle and there is no numeral correlation.

In most cases, it is simply unreasonable to assign absolute values to two independent variables. The nature of SPVS does not allow the arithmetic to give any indication of what the absolute orders might be (as discussed in the introduction). This all makes it difficult to say whether scribes used alignment to indicate the relative place values between numbers. For example, the second line of MS 3955 (Fig.29) is correct for $10°10' \times 9°$ and for $10°10' \times 9'$, or indeed any multiple of 60 thereof. This means it is impossible to say wither or not positioning the 9 just below the first 10 indicates that they are of the same order.

The consistent lack of diligent alignment, as seen in these examples, may seem startling to the modern viewer, given the exacting layout we employ for manual arithmetic and the apparent want for rigid arrangement due to the ambiguity of the SPVS. Indeed Friberg (2007) points out at there are certain "positional errors" that are almost inevitable without end zeros. However, consideration should be given to the standard practice of left-justifying lines (Robson, 2005). It seems that this remained the primary convention for numbers, rather than to align them to reflect relative place value. MS 3037 is at leat one example where this it blatantly manifest. In fairness, relative place alignment may have been a more common practice than can be established here, but the nature of SPVS in effect conceals the evidence within the arithmetic.

3.2 The 'Special Tables'

MS 3937 (Fig.30) and MS 3906 (Fig.31) are two Old Babylonian texts from the Schøyen Collection. They warrant a separate section, first because it is difficult to say exactly what type of text they are and second because they contradict many of the formatting and alignment observations made in this study.

In *'A Remarkable Collection of Babylonian Texts'* (2007), Friberg classifies these texts as 'Special Tables', and he introduces them by noting:

> There is no big difference between "clay tablets with squaring exercises" and "special tables of squares". Loosely speaking, the only difference is that a clay tablet with squaring exercises looks very much like a clay tablet with multiplication exercises, while a special table of squares looks like a standard table of squares.

Figure 26: YBC 1793, a tablet showing additions of weights of silver. (Robson, 2008)

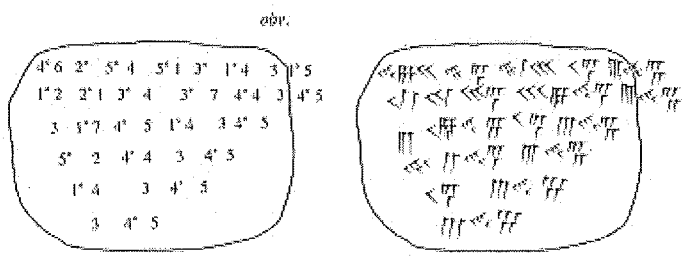

Figure 27: MS 2242, a tablet showing a factorization problem in which place alignment may be present. (Friberg, 2007)

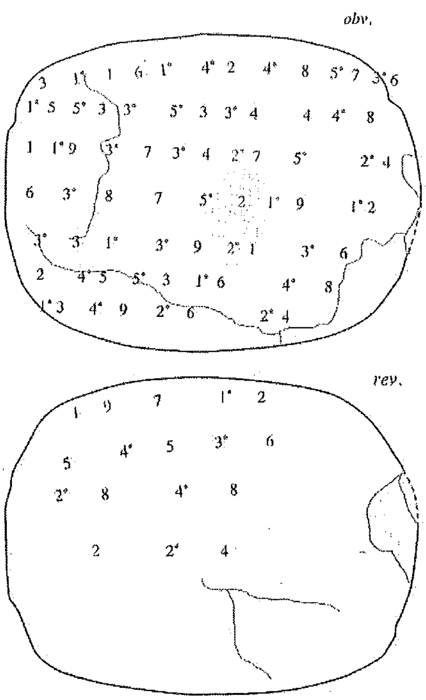

Figure 28: MS 3037, a tablet showing a factorization problem in which there is no alignment. (Friberg, 2007)

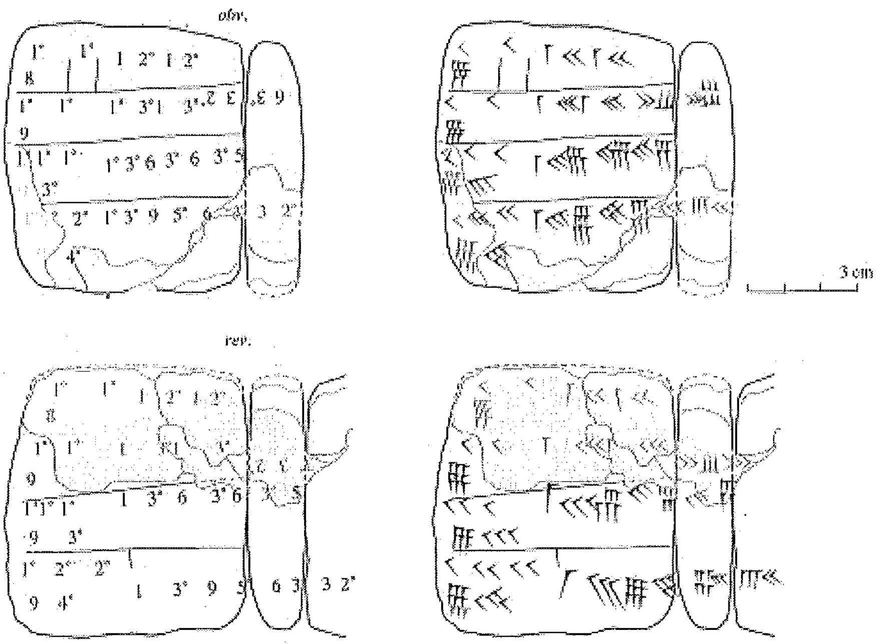

Figure 29: MS 3955, a tablet with the same multiplication exercises repeated on each side. (?)

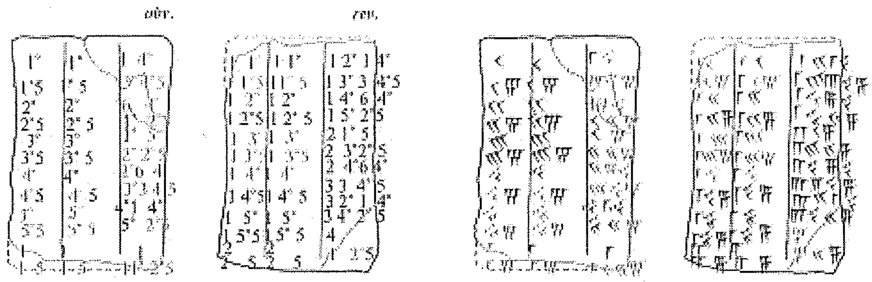

Figure 30: MS 3937, a 'table' that follows the straight multiplication format. (Friberg, 2007)

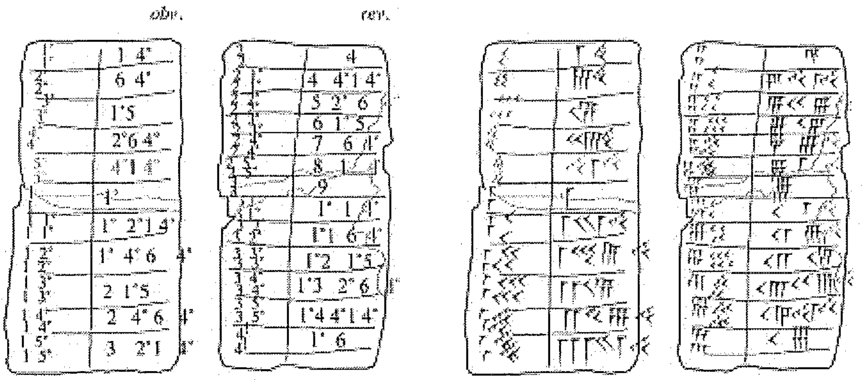

Figure 31: MS 3906, a 'table' that follows the triangular multiplication format. (Friberg, 2007)

It is true that an uncritical glance might suggest that these texts are tables; they both lists values and their squares in a list of increasing order in a manner that suggests data storage, and as neither have words and both are ruled they appear more 'tabular' to modern sensibilities than many other texts, including the numerical lists above. Yet closer examination shows that they are formatted in the manner of multiplication exercises (see §3.1.1). A single line of MS 3906 quite clearly appears as triangular multiplication, ruled horizontally and vertically in a manner that is not unusual for exercises tablets. MS 3937 is formatted in a straight multiplication style, with the product placed in line with the factors horizontally.

At this point, it may be asked whether the definition of a table should be based on the objective of the author and whether his intention was to store data for referral, or whether he was performing an operation for its own sake. These texts show a perplexing combination of the formats that have been consistently seen only in exercises and the list-like properties that we would expect from data-storage. Whichever way they are classified, they are exceptions to patterns that have been observed in previous sections. If they are tables, they are the only known third format tables of squares, and also the only third format tables that are known to be ruled vertically (§2.2.1). Indeed, they would contradict the stipulation that the multiplication formats only appear on rough arithmetic exercises and not in more formal texts (§3.1.1). If they are classed as arithmetic exercises, MS 3937 would be the only known straight multiplication that is ruled between factors (§3.3) and the only known example of straight multiplication that does not appears amid other types of calculations (§3.1.1).

The crux of this intriguing uniqueness is that both these texts hint at a certain mid-step in the cognitive leap from the use of words to explicitly state an operation, through the use of alignment that still requires the repeating of the factors, to the 'true tabulation' of expressing a function is its entirety though the alignment of input and output values and the separation of these set with a line.

3.3 Visual Features

Part of the study of alignment involves looking at the use of visual features such as lines and open spaces to accentuate alignment and clarify the relationships between numerals. Ruled lines can both separate and connect numbers. Dividing lines put numbers into separate classes and can help to define their meaning. Physical lines between numbers demonstrate their connection numbers even if they are not at right angles to each other. This is a regular feature of tables, where lines are an indication of operation, and it is interesting to see similar aspects in standard arithmetic exercises.

There are plenty of extant tablets that are ruled horizontally to separate lines, similarly to lines in an exercise book. Exercises that are formatted line-by-line, such as factorization and reciprocal exercises, are often ruled in this way, such as the likes of VAT 5457 (Fig.32) which has several un-used ruled lines. But the fact that horizontal rulings were not always used suggests that horizontal lines were a optional extra that was used purely to assist formatting rather than influence the meaning of the document. Lines such as these could apply to writing as much as to mathematics and have little effect on the overall understanding of the tablet. It is where lines interrupt or circumvent the mathematics that visual 'extras' and the alignment they accentuate gain more significance.

The different multiplication formats show differences in the use of visual features. Triangular multiplication shows various combinations of ruling, either horizontal or vertical, or both, or neither. Horizontal rulings were done to differentiate between problems (such as on

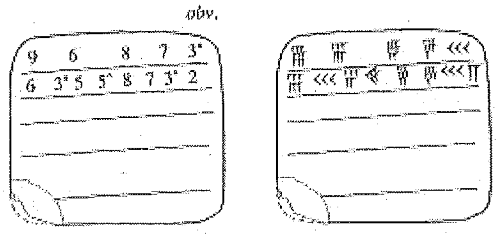

Figure 32: VAT 5457, a tablet showing a reciprocal exercise, including some numbers that were meant for multiplication. It features several un-used ruled lines. (Friberg, 2007)

MS 2729 (Fig.17)). There are no know examples of triangular multiplication being ruled on each line, between factors. This would possibly have interrupted the placement of the product. Certainly it seems is a reasonable explanation for no horizontal rulings being seen on continuous triangular aligned multiplication, as the zig-zag formatting wouldn't have allowed for it.

Where vertical rulings were done, they were between factors and products. Notably, there is pretty much always discernable separation factors and products in triangular multiplication; where there is no line, there tends to be a fairly sizable gaps separating the factors and products.

Straight multiplication does not tend to show clear visual separation.[6] There are no known examples of this format being ruled between factors and usually there is not even a perceivable gap. This makes it almost impossible to recognize two separate factors without back-calculating the equation, especially given that gaps naturally occur within cuneiform numbers. Take for example VAT 5457 (Fig.32), which is a reciprocal exercise, showing 9 6 8 and it's reciprocal (miscalculated as 6 35 58 7 30 when it should read 6 35 30 28 7 30). As Friberg (2007) points out, VAT 5457 can easily be read as 9 6 8 7 30 and 6 35 58 7 32, but in fact the 7 30 and 2 on the ends of the lines are actually meant for multiplication as intermediate steps in Friberg's 'trailing part algorithm'.

It should be noted that the rulings are not always very well respected. Often scribes slip over the pre-draw rulings, such as in the second line of MS 2728 (Fig.18) where the scribe has run out of room for 55. It is impossible to say definitively whither this is because the scribe is still learning or has no intention of staying within the boundaries. Yet the fact that such lines were used indicates that the alignment was important enough to warrant placing boundaries. It is interesting to see, in both multiplication formats, the consistent separation of exercise and answer - in triangular multiplication by using a line or a gap and in straight multiplication by putting it on the next line. From this perspective, it is possible to say that alignment was used to indicate the *presence* of an operation. Conversely, the lack of visual separation of factors, particularly in straight multiplication, suggests that indicating the nature of the operation was less important.

4 Conclusions

The original hypothesis of this study was that alignment in ancient near eastern mathematical texts could have performed the function of modern symbols. However, it can only be weakly

[6]For discussion of the possible exception, see section 3.2.

concluded that alignment did the work of symbols such as the $+$, $\times$, $\div$ and the decimal point.

Evidence that alignment was used to indicate the relative value of places between numbers is uncertain at best and at worst badly conceived. In the few cases where it is possible to assign units or guess at the absolute order of places there is minimal evidence of such a practice. In most cases, it is impossible to make a reasonable comment.

So far as operational alignment goes, there is evidence of a weak convention in rough-working texts. The two forms of multiplication, which here have been characterized as 'triangular' and 'linear', can be identified in many texts. Although layout of these exercises is not rigid, the evidence suggests that it was a standard practice. A far more flimsy argument may be made for other operations. Mostly, calculations were transformed into multiplication exercises and formatted as such, or hap-hazard alignment resulted.

The use of visual features, predominantly ruled lines, emphasizes these arrangements. Particularly, the general separation between factors and the product in multiplication exercises has been observed. This enhances the idea that while the presence of a calculation was emphasized, the nature of that calculation was such an important communication. This may simply have been because the scribe did not mean the tablet to be viewed by anyone but himself, or because it was fairly safe to assume that a calculation would be multiplication. However, it may also be an indication of a more fundamental attitude towards formatting.

This theory is backed up by the evidence of tables from the ancient near east. The rudimentary use of alignment in many arithmetical tables, or numerical lists, tends to clearly separate the 'input' values from the 'output' values. However, the explanation of the operation is usually done explicitly with words, rather than implicity with alignment. Sometimes, words are not used at all, as in the more uncommon third format arithmetical tables. In these texts, the presence of a relationship is fairly clear, due to the alignment of the columns, but the nature of the operation must be found by inspection and back-calculation.

Other, larger tables (the table compendiums and complex tables) suggest at a different aspect of the ancient near eastern mind-set: an emphasis on presentation of data and logic in a linear manner. There appears to have been a focus on the logic of each line reading as it would be read. Possibly, this mimics the prolificacy of written formats involving words. This is seen in Ash 1924.796, where the lines do not work in both directions, but instead read so that information is repeated. The existence of table compendiums also supports this view. Table compendiums had the potential to be presented as complex tables, but instead they were presented as 'lists that follow lists' rather than 'lists of lists'. This suggests that ancient near-eastern scribes were not comfortable with the idea of a table in which the columns refer back to a single 'root' column and where there were columns 'interrupting' the left-to-right logic of each line. For example, in a table of columns $[n \quad \frac{1}{n} \quad n^2]$, it seems that they were simply not comfortable with, or had never conceived, the presence of the unrelated $\frac{1}{n}$ when they were interested in n^2. The possibility that 'its' (as in, 'its square root') could refer to a number beyond the most immediate digit appears to be an alien one. There appears to have been an inherent desire for logic to work progressively along a line rather than working back on itself.

This, however, is exactly the approach that administrative tables show, and further study of the use of alignment in cuneiform culture would undoubtedly look more closely at this format. The likes of Plimpton 322 show that there was the potential for the sophistication and complexity of administrative tables to be used in paedological texts. At the very least, Plimpton 322 shows that while concepts of tabularization were uncomfortable for many scribes, they were not totally alien.

In *'Words and Pictures: New Light on Plimpton 322'* (2002), Eleanor Robson discusses the influences of conventions that mathematicians are not even aware of. This seems to be reflected in this study also; that there was no formal convention of the expression of operation through alignment, but it was common practice made it so that certain arrangements possibly had significance. That although there was great potential for tabularization to be used in mathematics, as evidenced by administrative tables and the 'cross-over' of formats evidenced in Plimpton 322, the habits of paedological teaching and learning remained. Although alignment was used, it was chiefly as a supplement to the primary practices of left-justification, using words for explicit communication and the line-wise flow of logic.

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Finally, I have enjoyed working with all my summer research buddies, but none more so than Rachel Gough. She has focused, distracted, provided tea, initiated press-up competitions (I won) and become a gold standard of commitment and cheerfulness. Rach, for making wyteboard markers a necessity and for everything else, you've been amazing.